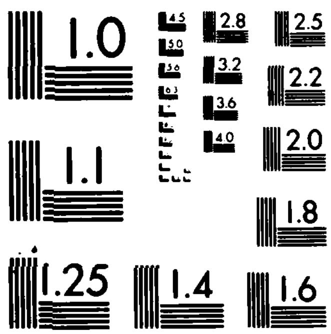
AD-A140 679 ON THE DISTRIBUTION OF COMPLEXITY FOR DE BRUIJN SEQUENCES(U) NAVAL POSTGRADUATE SCHOOL MONTEREY CA R L HOLDAHL JUN 83 1/1

UNCLASSIFIED F/G 12/1 NL



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS 1963-A

NAVAL POSTGRADUATE SCHOOL
Monterey, California

THESIS

ON THE DISTRIBUTION OF COMPLEXITY
FOR DE BRUIJN SEQUENCES

Robert LaVern Holdahl

June 1983

Thesis Advisor: H. Fredricksen

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER	2. GOVT ACCESSION NO. A140679	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) On the Distribution of Complexity for de Bruijn Sequences		5. TYPE OF REPORT & PERIOD COVERED Master's Thesis; June 1983
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Robert LaVern Holdahl		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Postgraduate School Monterey, California 93940		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Naval Postgraduate School Monterey, California 93940		12. REPORT DATE June 1983
		13. NUMBER OF PAGES 71
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report)
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

de Bruijn
complexity

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Binary sequences have had application in communication systems for many years. Shift registers have been used in their generation, because of the ease and economy of their operation. For certain applications, nonlinear feedback functions are used by shift registers of span n to generate sequences of lengths up to 2^n.

Block #20 Contd.

The sequences of maximum length 2^n and their generation are the subject of this thesis. In particular the ways of generating these sequences using nonlinear feedback shift registers and their correlation to linear feedback shift registers are described. Complexity is the term given to the length of the shortest linear feedback shift register generating a maximum length 2^n sequence.

Games and Chan [Ref. 1] have given considerable study to the subject of complexity. Some of the problems they left are discussed further in this paper. It will be shown that the complexity of a de Bruijn sequence (S) is the same as the complexity of its reverse (r S), complement ($\overline{S}$) and its reverse complement (r $\overline{S}$). Sequences (S) for which r S = $\overline{S}$ are termed RC sequences. It is shown that RC sequences exist for every odd $n>3$. In addition a lower bound will be established for the number of RC sequences occurring for each odd n.

On the Distribution of Complexity for de Bruijn Sequences

by

Robert LaVern Holdahl
Major, United States Marine Corps
B.S., South Dakota State University, 1972

Submitted in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE IN APPLIED MATHEMATICS

from the

NAVAL POSTGRADUATE SCHOOL
June 1983

Author: Robert L. Holdahl

Approved by: ______________________
Thesis Advisor

____
Chairman, Department of Mathematics

____
Dean of Information and Policy Sciences

ABSTRACT

Binary sequences have had application in communication systems for many years. Shift registers have been used in their generation, because of the ease and economy of their operation. For certain applications, nonlinear feedback functions are used by shift registers of span n to generate sequences of lengths up to 2^n.

The sequences of maximum length 2^n and their generation are the subject of this thesis. In particular the ways of generating these sequences using nonlinear feedback shift registers and their correlation to linear feedback shift registers are described. Complexity is the term given to the length of the shortest linear feedback shift register generating a maximum length 2^n sequence.

Games and Chan [Ref. 1] have given considerable study to the subject of complexity. Some of the problems they left are discussed further in this paper. It will be shown that the complexity of a de Bruijn sequence (S) is the same as the complexity of its reverse (r S), complement ($\bar{S}$), and its reverse complement (r $\bar{S}$). Sequences (S) for which r S = $\bar{S}$ are termed RC sequences. It is shown that RC sequences exist for every odd $n \geq 3$. In addition a lower bound will be established for the number of RC sequences occurring for each odd n.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

I. INTRODUCTION

A. APPLICATION

de Bruijn sequences have been the subject of recurring interest since their discovery over a century ago. Recent developments in radar ranging [Ref. 2: Ch. 13], error correcting codes, secure or limited access code generators [Ref. 3: pp. 12-14], and mathematical modeling have made application of de Bruijn sequences because of their properties. The binary shift registers readily model digital computers with electronic states where "1" indicates the on state and "0" indicates the off state.

B. DEFINITION OF de BRUIJN SEQUENCES

For a positive integer n, a de Bruijn sequence of span n is a complete binary cycle of length 2^n which is a sequence $S = \{s_1, s_2, \cdots, s_{2^n}\}$ taken in circular order (s_1 follows s_{2^n}) such that all possible n-tuples occur exactly once. [Ref. 4: p. 120]. Complete binary cycles are often called de Bruijn sequences after the Dutch mathematician, N. de Bruijn, who in 1946 proved the existence of complete binary cycles having length 2^n and that they numbered $2^{2^{n-1}-n}$. de Bruijn was preceeded by Flye Saint-Marie a half century earlier in an obscure publication [Ref. 5]. As an example for n = 3, Table 1.1 shows both ($2^{2^{3-1}-3}$) de Bruijn sequences of length eight (2^3). Both of these sequences are examined to confirm that each of the eight 3-tuples occur exactly once. Notice that 2 elements of the sequence are repeated as signified by the bar ($\overline{00}$), which was necessary on each sequence to complete the last two 3-tuples.

TABLE 1.1

3-TUPLE COMPOSITION OF de BRUIJN SEQUENCES OF SPAN 3

```
0001110100̅0̅          0001011100̅0̅
000                   000
 001                   001
  011                   010
   111                   101
    110                   011
     101                   111
      010                   110
       100                   100
```

There are 2^n different possible starting points for each de Bruijn sequence; however, each cyclic permutation is considered equivalent. It is often a matter of convenience to start each sequence in a canonical way with the all zero or all one n-tuple. This reduces confusion when working with a number of different de Bruijn sequences.

C. PSEUDO-RANDOMNESS

Since each binary n-tuple occurs exactly once on the cycle, the sequence models a uniform distribution with the probability of given n-tuple occurring to be exactly $1/2^n$. The sequence is not entirely random, however, since each n-tuple ($a_1\ a_2 \cdots a_n$) has but two possible successors ($0\ a_1\ a_2 \cdots a_{n-1}$ or $1\ a_1\ a_2 \cdots a_{n-1}$). de Bruijn sequences satisfy the randomness properties of equidistribution of 0's and 1's and run lengths [Ref. 3: p. 10] which would be expected from the tossing of a fair coin. Thus, de Bruijn sequences have a pseudorandom property.

D. K-ARY SEQUENCES

In general it is possible to work with k possible states for each position of an n-tuple. Flye Saint-Marie in 1894, showed the existence and determined the number of complete cycles for n-tuples of k characters to be $[(k-1)!]^{k^{n-1}} k^{k^{n-1}-n}$. [Ref. 5]. This report is restricted to the binary case ($k = 2$) due to its principal application in electronic communications and computers.

E. DEFINITIONS

For $S = \{s_1\ s_2 \cdots s_k\}$ where $s_i \in \{0,1\}$ for $1 \le i \le k$ the following are defined:

1. Let W(S) denote the weight of S.

$$W(S) = \sum_{i=1}^{k} s_i$$

2. Let P(S) denote the parity of S.

$$P(S) = \sum_{i=1}^{k} s_i \pmod 2$$

3. Let $l(S)$ denote the length of S.

$l(s) = k$ where k is the number of positions in S.

II. SEQUENCE GENERATORS AND OPERATORS

A. SHIFT REGISTERS

In practice shift registers are used to generate de Bruijn sequences. An n-stage shift register has n memory registers (x_1, x_2, $\cdots$, x_n) which shift their contents (0 or 1) to the next register upon command. For example, in Fig. 2.1, the contents of x_1 will transfer to x_2, x_2 to x_3, etc., at the appropriate time with X_n serving as the output. However, the contents of an n-stage shift register would empty in n shifts if no input was provided to x_1.

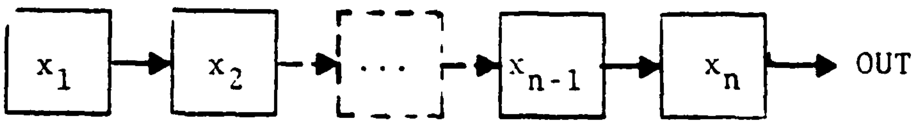


Fig. 2.1 n-Stage Shift Register.

B. FEEDBACK SHIFT REGISTERS

Feedback networks are added to provide an input to the 1st stage of the shift register. Note that in practice the output can be taken from any stage of the shift register. In fact, the contents of any stage is the same as the contents of any stage is the same as the contents of the n^{th}

stage, merely shifted by a certain amount. In Fig. 2.2 the feedback function f is added to the n-stage shift register to generate non-trivial sequences. The coefficients $a_0, a_1, \cdots, a_n$ are 0 or 1 with $a_0 = a_n = 1$.

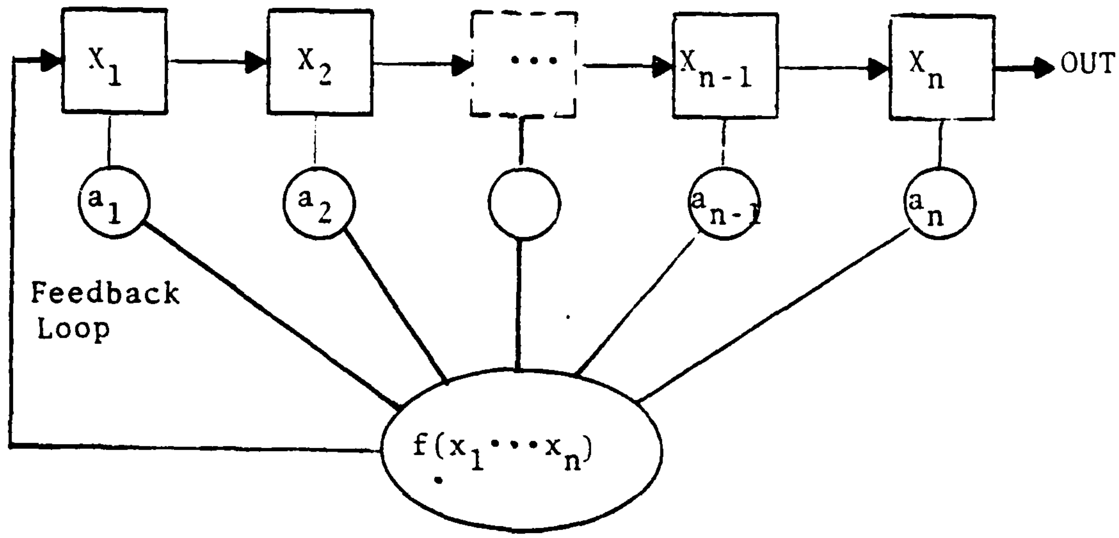


Fig. 2.2 n-Stage Feedback Shift Register.

1. <u>Linear Feedback Functions</u>

A feedback function f is classified as linear if the function $f(x_1, x_2, \cdots, x_n)$ is restricted to be of the form $f = \sum_{i=1}^{n} a_i x_i$ where the addition is modulo 2 addition. The addition (mod 2) table is given in Table 2.1 below. Note that subtraction is equivalent to addition when operating modulo 2.

TABLE 2.1

ADDITION (MODULO 2)

$\oplus$	0	1
0	0	1
1	1	0

2. Nonlinear Feedback Functions

If the feedback function $f(x_1, \cdots, x_n)$ utilizes multiplication (mod 2) and addition (mod 2) then the feedback function is nonlinear. (We also say that the linear feedback functions are vacuously nonlinear feedback functions.) Modulo 2 multiplication is identical to multiplication in the integers with the restriction of the alphabet used being the set {0, 1} as described in Table 2.2.

TABLE 2.2

MULTIPLICATION (MODULO 2)

$\cdot$	0	1
0	0	0
1	0	1

3. Comparison of Linear and Nonlinear Feedback Functions

Considerable research has been devoted to the study of linear feedback functions because of their ease of analysis. Nonlinear feedback functions are much more complicated.

However, the class of sequences that can be generated by nonlinear functions is much greater than for linear functions.

The number of binary maximum length (2^n-1) linear sequences that can be generated by a n-stage linear feedback shift register (LFSR) is approximately $\frac{2^n}{n}$ [Ref. 7]. The reason that the maximum length linear sequences only have length 2^n-1 is that the zero state is a fixed point under the linear function. This poses no problem since the maximal length linear sequences are readily made into de Bruijn sequences having length 2^n simply by adding a zero to the (n-1)-tuple of zeros. The feedback function involved then becomes a nonlinear function.

By comparison the number of binary nonlinear full length sequences (2^n) that can be generated by a nonlinear feedback shift register (NFSR) is exactly $2^{2^{n-1}-n}$. Below, Table 2.3 compares these numbers for $3 \leq n \leq 8$. The number of maximum length sequences will be of interest later in Section III, where the concept of complexity in generating these sequences will be developed. Various algorithms are listed in a survey by Fredricksen [Ref. 5] for generating de Bruijn sequences.

C. SUCCESSOR AND PREDECESSOR STATES

The contents of an n-stage shift register at a specific time is usually referred to as its "state". As mentioned in Section I when discussing the pseudorandomness of de Bruijn

TABLE 2.3

NUMBER OF MAXIMUM LENGTH SEQUENCES

n	Linear	Nonlinear
3	2	2
4	4	16
5	6	2048
6	6	67,108,864
7	18	1.4×10^{17}
8	30	2.7×10^{36}

sequences each state has two possible successor (conjugate) states depending on whether the feedback function generates a "0" or "1" as the input to register x_1. Likewise each state has two predecessor (companion) states depending on whether the previous output was a "0" or "1". The adjacency quadruple in Fig. 2.3 centers around then (n-1)-tuple $x_2 \cdots x_n$. That is, each input state has two possible successors depending on whether the feedback is 0 or 1, and each output state has two possible predecessors depending on whether the previous output from the n^{th} stage was 0 or 1.

In Table 2.4 we show an example for n = 3 of the possible successors and predecessors for each state.

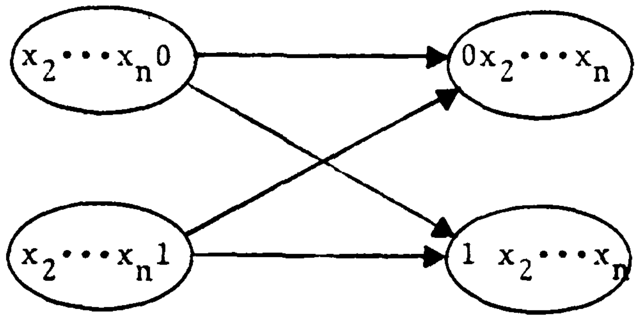


Fig. 2.3 Adjacency Quadruple.

TABLE 2.4

CONJUGATE AND COMPANION STATES (n=3)

Predecessors	State	Successors
000-001	000	000-100
010-011	001	000-100
100-101	010	001-101
110-111	011	001-101
000-001	100	010-110
010-011	101	010-110
100-101	110	011-111
110-111	111	011-111
Output		Input

D. de BRUIJN DIAGRAMS

The de Bruijn diagram compactly contains all the above information. The diagram contains 2^n vertices corresponding to the 2^n states and two directed edges from each state to the possible pair of successor vertices. As a result two arrows exit every vertex leading to conjugate states, and two arrows enter every vertex arriving from companion states. For an example consider the de Bruijn diagram G_n for n = 3 in Fig. 2.4. An Eulerian path through G_n is defined as a

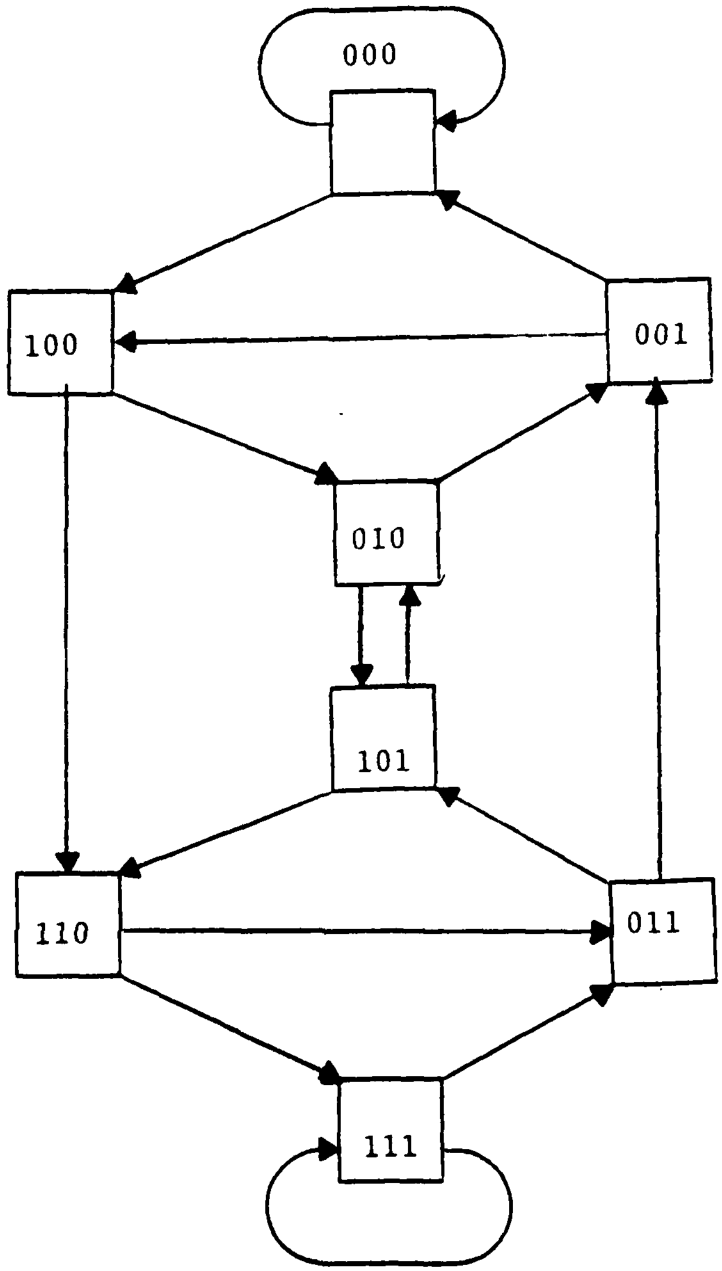


Fig. 2.4 Graph G_3.

path which visits each edge exactly once. A path which visits every vertex of G_n is called a Hamiltonian path. A Hamiltonian path in G_n is a de Bruijn sequence of span n. If the edges of G_n are labeled with an n + 1-tuple defined by the labels on the predecessor and successor states, then the Eulerian paths in G_n correspond to Hamiltonian paths--and de Bruijn sequences in the graph G_{n+1}. The Hamiltonian path in Fig. 2.5 is a subgraph of G_3 in Fig. 2.4 with exactly one edge emanating from each vertex.

E. OPERATORS

For $S = s_1 s_2 \cdots s_{n-1} s_n$ where $s_i \in \{0,1\}$ for $1 \leq i \leq n$ the following operators are defined.

1. <u>Identity Operator</u>

The identity operator (e) operating on S denoted $eS = s_1 s_2 \cdots s_{n-1} s_n = S$.

2. <u>Reverse Operator</u>

The reverse operator (r) operating on S is denoted r S, where $r\,S = s_n s_{n-1} \cdots s_2 s_1$. The reverse operator preserves the weight, parity and length of S. Note also that $(r)^2 \equiv e$.

3. <u>Complement Operator</u>

The complement (dual) operator on S is denoted d S or $\overline{S}$, where $\overline{S} = \overline{s}_1 \overline{s}_2 \cdots \overline{s}_{n-1} \overline{s}_n$ such that $\overline{s}_i = s_i \oplus 1 \pmod 2$ for $1 \leq i \leq n$.

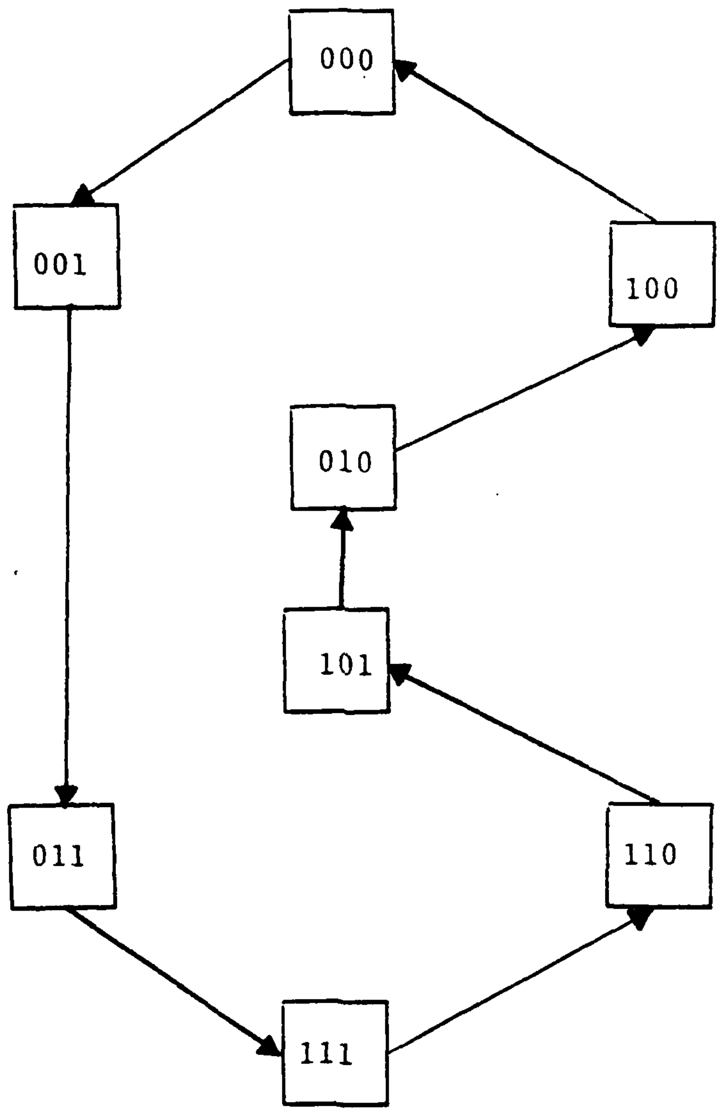


Fig. 2.5 A Hamiltonian Path through G_3.

Lemma 1: If $W(S) = k$ for $S = s_1 s_2 \cdots s_n$ then $W(\bar{S}) = n-k$.

Proof: $W(S) = k$ implies k nonzero elements εS and $n-k$ zero elements εS.

Complementing $n-k$ zero elements produces $n-k$ nonzero elements $\varepsilon \bar{S}$.

$\therefore W(\bar{S}) = n-k$

Q.E.D.

Lemma 2: If $P(S) = a$ where $a \varepsilon \{0,1\}$ and $S = \{s_1 s_2 \cdots s_n\}$, then $P(\bar{S})$ is a iff n is even.

Proof: $P(S) = \sum_{i=1}^{n} s_i \pmod 2 = a$

$$P(\bar{S}) = \sum_{i=1}^{n} \bar{s}_i \pmod 2 = \sum_{i=1}^{n} s_i \oplus 1 = a \oplus n$$

therefore $P(\bar{S}) = P(n-a)$

Q.E.D.

Also note $(d)^2 = e$, and that the reverse and complement operators are commutative, i.e. $rd = dr$.

4. Reverse Complement Operator

The reverse complement operator on S is denoted $r\bar{S} = \bar{s}_n \bar{s}_{n-1} \cdots \bar{s}_2 \bar{s}_1$. Weight and parity are effected by the reverse complement operator exactly as with the complementation operator.

These operators apply to sequences as well, since sequences are composed of n-tuples. Table 2.6 shows the effect of operators on de Bruijn sequences of span 3 and 4 where r S, $\overline{S}$, and r $\overline{S}$ are all de Bruijn sequences, as they always are if S is a de Bruijn sequence.

TABLE 2.5

de BRUIJN SEQUENCE OPERATORS

<u>n = 3</u>			(Canonical Form)
S =	00011101	=	00011101
rS =	10111000	=	00010111
$\overline{S}$ =	11100010	=	00010111
r$\overline{S}$ =	01000111	=	00011101

<u>n = 4</u>			(Canonical Form)
S =	0000111101001011	=	0000111101001011
rS =	1101001011110000	=	0000110100101111
$\overline{S}$ =	1111000010110100	=	0000101101001111
r$\overline{S}$ =	0010110100001111	=	0000111100101101

If as in the case for n = 3 in Table 2.5 a sequence S is equivalent to its reverse complement, then S is termed a RC sequence. This applies to both de Bruijn and non-de Bruijn sequences. The following theorem appears in a paper by Etzion and Lempel [Ref. 6].

Theorem 1: A sequence S is a RC sequence if ℓ (S) is even and S = {X, r$\overline{X}$} for some X where X is a binary string.

F. *LINEAR RECURSION FORMULA*

de Bruijn sequences generated by a nonlinear function can also be generated by a linear function $f\ (x_1 \cdots x_n) = x_o$ having the linear recursion formula

$$\sum_{j=0}^{n} a_j\ x_j = 0\ .$$

Here each a_j determines a tap to the j^{th} stage of the register. The linear generator polynomial must be $(x + 1)^k$ for some positive integer k, due to the periods of de Bruijn sequences (2^n) and results from theorems given by Golomb [Ref. 7: pp. 27-43].

1. Pascal Triangle (Mod 2)

The values for each a_j are summarized in the Pascal triangle (mod 2) Table 2.6 for various values of n. They are the coefficients generated by the binomial expansion of $(x + 1)^k$ mod 2.

TABLE 2.6

PASCAL TRIANGLE (MOD 2)

k = 0						1						
1					1		1					
2				1		0		1				
3			1		1		1		1			
4		1		0		0		0		1		
5	1		1		0		0		1		1	$(x+1)^5$
	a_o		a_1		a_2		a_3		a_4		a_5	

2. Pascal Triangle Properties

(a) The a_j sequence is palindromic, that is $a_o = a_n = 1$, $a_1 = a_{n-1}$, $a_2 = a_{n-2}$, etc.

(b) The number of nonzero coefficients is even, that is $\sum_{j=0}^{n} a_j = 0 \pmod 2$.

III. COMPLEXITY OF de BRUIJN SEQUENCES

A. DEFINITION OF COMPLEXITY

The term complexity indicates a general measure of the predictability of a sequence. Various definitions of complexity are in use, but this paper will use the one given by Chan and Games [Ref. 1]. Their definition of complexity is the length of the shortest LFSR required to generate a sequence. They apply their results primarily to de Bruijn sequences. This definition is also the one used by Herlestom [Ref. 8] to look at two different shift register cycle generators. For a de Bruijn sequence S, let C(S) denote the complexity of S.

Previous results by Chan and Games [Ref. 1] establish lower and upper bounds of $2^{n-1}+n$ and 2^{n-1} respectively for the complexity of a de Bruijn sequence. Thus, the length of a LFSR is almost as long the sequence as opposed to a n-stage NFSR which generates the sequence. Table 3.1 serves as a ready reference for the upper and lower limits of complexity for $3 \leq n \leq 8$. The upper bound is known to be attained for all n and the lower bound for all $n \leq 6$.

B. COMPLEXITY ALGORITHM

A fast method for determining the complexity of a de Bruijn sequence was developed by Games and Chan [Ref. 9]

TABLE 3.1

LIMITS OF COMPLEXITY

Span (n)	Lower Bound $(2^{n-1}+n)$	Upper Bound $= (2^n-1)$
3	7	7
4	12	15
5	21	31
6	38	63
7	71	127
8	136	255

and is represented by the flow chart in Fig. 3.1. Let S be a de Bruijn sequence then $\ell(S) = 2^n$. Further let $S = A:B = D$ where $A = \{a_1\ a_2 \cdots a_{2^{n-1}}\}$ and $B = \{b_1\ b_2 \cdots b_{2^{n-1}}\}$ then $D = \{a_1\ a_2 \cdots a_{2^{n-1}}\ b_1\ b_2 \cdots b_{2^{n-1}}\}$. To serve as a guide an example for n = 5 is now presented. Let the de Bruijn sequence S = {11111000001000110101100101001110} having length $\ell = 32$.

A	1111100000100011	ℓ = 16
B	0101100101001110	C = 0
A+B	1010000101101101	C = 16

A	10100001	ℓ = 8
B	01101101	C = 16
A+B	11001100	C = 24

A	1100	$\ell = 4$	Since $A \oplus B = 0$
B	1100	C = 24	
A+B	0000	C = 24	C is unchanged
A	11	$\ell = 2$	
B	00	C = 24	
A+B	11	C = 26	
A	1	$\ell = 1$	Add 1 since $\ell = 1$
B	1	C = 26	& A = 1
A+B	0	C = 26	+ 1 = 27

C. COMPLEXITY DISTRIBUTION

The nature of the distribution of complexity of de Bruijn sequences is the primary interest of this paper. Let $\alpha(c,n)$ denote the number of de Bruijn sequences having complexity c and span n. The complexity distribution of de Bruijn sequences of span n for $3 \leq n \leq 6$ are listed in Table 3.2. Unfortunately the large number of de Bruijn sequences for $n \geq 7$ (2^{57}) does not allow an exhaustive examination of their complexities.

Examination of this data led Chan and Games to conjecture that for $n \geq 3$ $\alpha(c,n)$ is congruent to 0 modulo 4.

D. EFFECTS OF OPERATORS ON COMPLEXITY

Chan et al. [Ref. 1] proved that $\alpha(c,n)$ for $n \geq 3$ was congruent to 0 modulo 2. By showing that for each de Bruijn

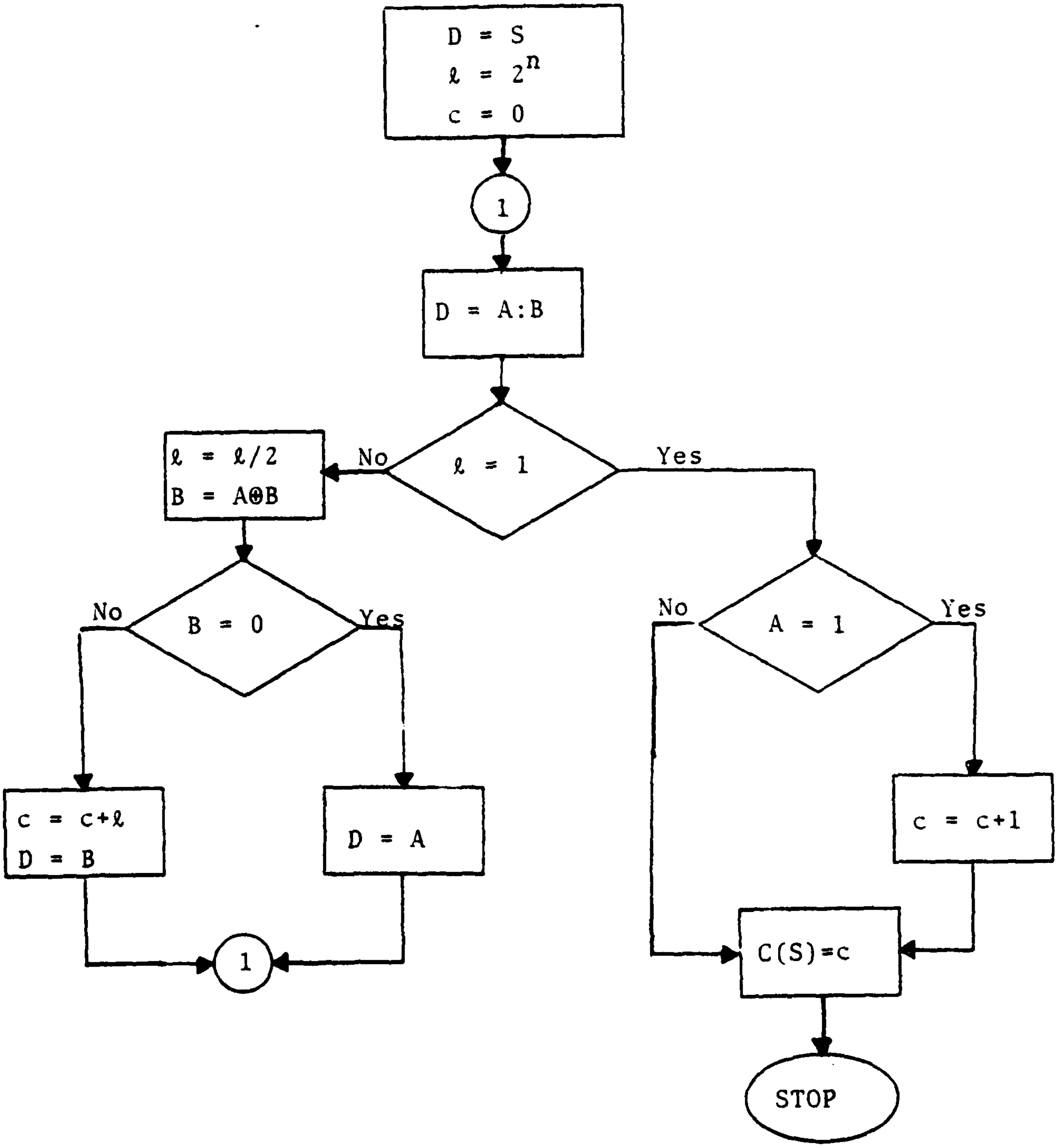


Fig. 3.1 Flowchart for Complexity Algorithm.

TABLE 3.2

COMPLEXITY DISTRIBUTION

n = 4

c	α(c,4)
12	4
13	0
14	4
15	8

n = 5

c	α(c,5)	c	α(c,5)
21	8	27	64
22	0	28	180
23	12	29	224
24	20	30	448
25	32	31	1024
26	36		

n = 6

c	α(c,6)	c	α(c,6)
38	448	51	8704
39	0	52	18096
40	32	53	34224
41	96	54	67700
42	160	55	126592
43	80	56	259320
44	432	57	519752
45	288	58	1041252
46	896	59	2090716
47	1168	60	4162352
48	2772	61	8342176
49	2352	62	16692832
50	5224	63	33731200

sequence having complexity c, there exists another different de Bruijn sequence $\bar{S}$ also having complexity c. Consequently they showed that the complement operator does not change the complexity of a de Bruijn sequence. Using a different approach, the next theorem will show that the complexity of a de Bruijn sequence is also the complexity of the de Bruijn sequences defined by the reverse operator and by the reverse complement operator as well.

Lemma 3. For $S = \{s_1\ s_2 \cdots s_{2^n-1}\}$, a de Bruijn sequence, $C(S) = c$ iff $\sum_{i=0}^{c} a_i\ s_i = 0$ for every s_i where each a_i comes from the c^{th} row of the Pascal triangle (mod 2) and c is the smallest integer such that the above recursion holds.

Proof. S is de Bruijn with $C(S) = c$ iff $s_i = f(s_{i+1} \cdots s_{i+c})$ for every $s_i \in S$, which is *equivalent to* $s_i = \sum_{j=1}^{c} a_j\ s_{i+j}$ from results by Golomb [Ref. 5: pp. 27-43] mentioned earlier. In turn $s_i = \sum_{j=0}^{c} a_j\ s_{i+j} = 0$ from properties of the Pascal triangle given in Section II. Q.E.D.

This lemma is used in the following theorem to show that complexity is preserved by various operators on de Bruijn sequences.

Theorem 2. If S is a de Bruijn sequence having complexity C, then $C(S) = C(\bar{S}) = C(rS) = C(r\bar{S}) = c$.

Proof. Part I (reverse operator)

Let $s_i\ s_{i+1} \cdots s_{i+c}$ be an arbitrary (c+1) long segment of S which satisfies

(1) $\sum_{j=0}^{c} a_j \, s_{i+j} = 0$

by Lemma 3. Since the a_j sequence is palindromic as shown earlier, $s_{i+c} \cdots s_{i+1} \, s_i$ a (c+1) long segment of rS satisfies equation (1). Thus C (rS) = c.

Part II (complement operator) substituting $\bar{s}_i \cdots \bar{s}_{i+c}$, the c+1 long segment of $\bar{S}$, into equation (1), the results are $a_o \, \bar{s}_i + a_1 \, \bar{s}_{i+1} + \cdots + a_{c-1} \, \bar{s}_{i+c-1} + a_c \, \bar{s}_{i+c} =$

$a_o \, (s_i + 1) + a_1 \, (s_{i+1} + 1) + \cdots + a_{c-1} \, (s_{i+c-1} + 1) + a_c$

$(s_{i+c} + 1) = \sum_{j=0}^{c} a_j \, s_{i+j} + \sum_{j=0}^{c} a_j$ (mod 2). Since there are an even number of nonzero a_js, $\sum_{j=0}^{c} a_j = 0$ and equation (1) is satisfied. Therefore, by Lemma 3 C($\bar{S}$) = c.

Part III (reverse complement operator), by Parts I and II, it follows immediately that C (r$\bar{S}$) = c. Thus C (S) = C (rS) - C ($\bar{s}$) = C (r$\bar{S}$)

Q.E.D.

IV. TRUTH TABLES

Linear and nonlinear feedback functions were used in conjunction with n-stage shift registers to generate de Bruijn sequences in Section II. The input x_0 was generated as some function f of the current state $x_1\ x_2 \cdots x_n$, that is

(1) $x_0 = f\ (x_1\ x_2 \cdots x_n)$

or equivalently writing f as in (2) when specializing for cycles only functions.

(2) $x_0 = x_n + g\ (x_1 \cdots x_{n-1})$.

Extensive use is made of equation (2) since results for cycles only are desired.

A. FULL TRUTH TABLE

The full truth table is a useful way of listing the functional value x_0 from equation (1) for each possible state of the shift register. The functional values in Table 4.1 is an example of a full truth table for a de Bruijn sequence (00011101).

Examining the full truth table it becomes apparent that the information in the lower half (below the dashed line) is redundant, since the values of $f\ (x_1\ x_2\ x_3)$ for $x_3 = 1$ in the lower half are merely the complements of the values of $f\ (x_1\ x_2\ x_3)$ for $x_3 = 0$ in the upper half for the identical values of $x_1\ x_2$. This suggests that the upper half (or the

TABLE 4.1

FULL TRUTH TABLE OF (00011101)

Current State $(x_3\ x_2\ x_1)$	Functional Value $x_0 = f\ (x_1\ x_2\ x_3)$
0 0 0	1
0 0 1	1
0 1 0	0
0 1 1	1
1 0 0	0
1 0 1	0
1 1 0	1
1 1 1	0

lower half) of the truth table can display all the information for a function generating only cycles.

B. HALF TRUTH TABLE

Focusing on equation (2), since x_n is always 1 in the lower half this can be viewed as complementing the functional values for g $(x_1 \cdots x_{n-1})$ in the upper half. In this manner the half truth table in Table 4.2 is constructed for the same de Bruijn sequence (00011101) to aid comparison with Table 4.1. From this point on the term truth table will always mean the upper half truth table with $x_n = 0$, unless noted otherwise.

TABLE 4.2

HALF TRUTH TABLE OF (00011101)

x_2	x_1	$g\ (x_1\ x_2)$
0	0	1
0	1	1
1	0	0
1	1	1

The string of functional values in the truth table is defined to be the generator $G = \{g_0\ g_1\ \cdots\ g_{2^{n-1}-1}\}$ where the subscripts of g are the decimal equivalent of $x_{n-1}\ \cdots\ x_1$. Both the weight and parity of generator G will be of interest as the investigation of this paper continues. Whenever the parity or weight of the truth table is mentioned in this paper, it is referring to the generator of that truth table.

C. ZERO TRUTH TABLE

The zero truth table is defined as the truth table in which all functional values are zero. The zero truth table for n = 3 contained in Table 4.3 is basically the output of a circulating register which generates pure cycles.

Pure cycles are defined as cyclic permutations of the original state. The zero truth table in Table 4.3 generates the four pure cycles shown in Fig. 4.1. The number of pure

TABLE 4.3

ZERO TRUTH TABLE FOR n = 3

x_2	x_1	$g(x_1\ x_2)$
0	0	0
0	1	0
1	0	0
1	1	0

cycles Z_n has been shown [Ref. 7: p. 120] to be

$$Z_n = \frac{1}{n} \sum_{d|n} \phi(d)\ 2^{n/d}.$$

Eulers totient function $\phi(d)$ is the number of fractions of the form $^a/d$ where $1 \le a \le d$ and $^a/d$ is in lowest terms. The summation is over all positive integers d which divide n, denoted $(d|n)$. It is further shown [Ref. 5] that Z_n-1 is the minimum possible weight truth table that can generate a de Bruijn sequence.

In the next section, the proof of theorem 6 gives an example of how pure cycles can be joined in a way to create a de Bruijn sequence that has a minimum weight truth table. Table 4.5 gives a listing of Z_n and Z_n-1 for $1 \le n \le 7$.

D. ONES TRUTH TABLE

The ones truth table assigns a functional value of 1 to every entry of $g(x_1 \cdots x_{n-1})$. The ones truth table depicts the output of a complementing shift register. The input is

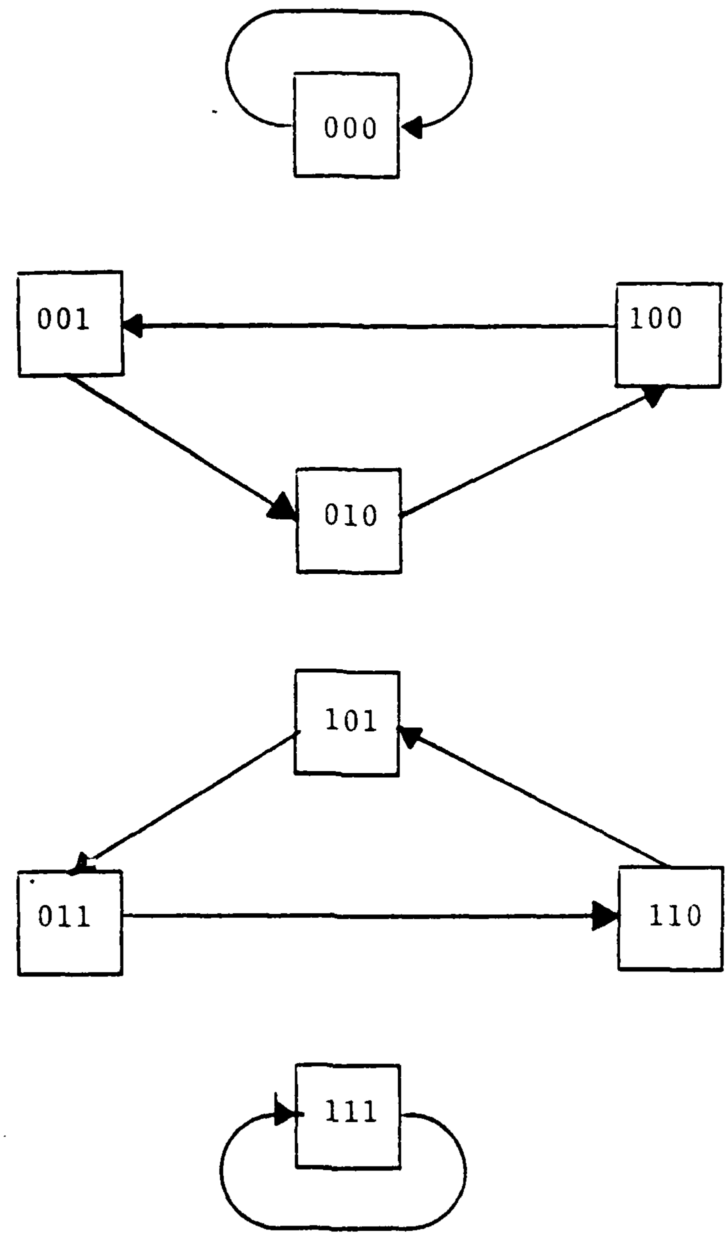


Fig. 4.1 Pure Cycles (n=3).

always the complement of the output, i.e. $f(x_1 \cdots x_n) = 1 + x_n$. The ones truth table for n = 3 is shown in Table 4.4. The complementing cycles from Table 4.4 are depicted in Fig. 4.2.

TABLE 4.4

ONES TRUTH TABLE (n = 3)

x_2	x_1	$g(x_1\ x_2)$
0	0	1
0	1	1
1	0	1
1	1	1

The number of complementing cycles Z_n^* [Ref. 7] is

$$Z_n^* = Z_{\frac{n}{2}} - \frac{1}{2n} \sum_{2d|n} \phi(2d)\, 2^{\frac{n}{2d}}.$$

Since summation for Z_n^* is over all 2d (even) numbers which divide n, for n odd $Z_n^* = Z_{\frac{n}{2}}$.

Accordingly Fredricksen [Ref. 5] shows truth tables of max weight that can generate a de Bruijn sequence is

$2^{n-1} - Z_n^* + 1$.

The values of Z_n^* and $2^{n-1} - Z_n^* + 1$ are given in Table 4.5 for $1 \le n \le 7$. Also included are the number of de Bruijn sequences having truth tables of maximum or minimum weight from a listing by Fredricksen [Ref. 5].

TABLE 4.5

PURE CYCLE COMPILATION

n	Z_n	$Z_n - 1$	# de Bruijn Sequences with Min Wt Truth Table	Z_n^*	$2^{n-1} - Z_n^* + 1$	# de Bruijn Sequences with Max Wt Truth Table
1	2	1	1	1	1	1
2	3	2	1	1	2	1
3	4	3	2	2	3	2
4	6	5	12	2	7	3
5	8	7	$2^6 \cdot 3^2$	4	13	2^6
6	14	13	$2^{14} \cdot 3^4 \cdot 5^2$	6	27	2^{14}
7	20	19	$2^{28} \cdot 3^5 \cdot 5^3 \cdot 13$	10	55	$2^{26} \cdot 3$

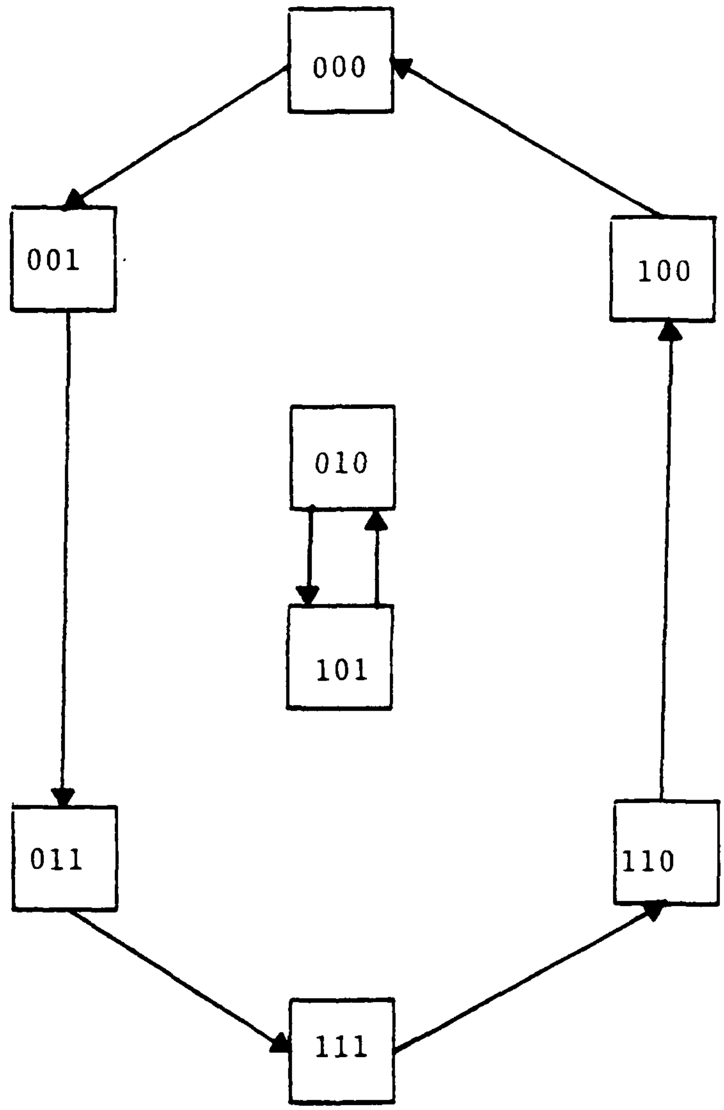


Fig. 4.2 Complementing Cycles (n=3)

E. EFFECTS OF CHANGING THE TRUTH TABLE

If the ones truth table in Table 4.4 is changed so that g (0,1) = 0 vice 1, the truth table of Table 4.6 is produced.

TABLE 4.6

TRUTH TABLE (TABLE 4.4 MODIFIED)

x_2 x_1	g $(x_1\ x_2)$
0 0	1
0 1	0
1 0	1
1 1	1

This change in the truth table effects the cycles of Fig. 4.2 by joining them as depicted in Fig. 4.3. Indeed the change produces a Hamiltonian path for the de Bruijn sequence (00010111).

Suppose the truth table of Table 4.6 is now changed so that g (00) = 0 vice 1. This produces the truth table in Table 4.7, and its cyclic composition in Fig. 4.4.

TABLE 4.7

TRUTH TABLE (TABLE 4.6 MODIFIED)

x_2 x_1	g $(x_1\ x_2)$
0 0	0
0 1	0
1 0	1
1 1	1

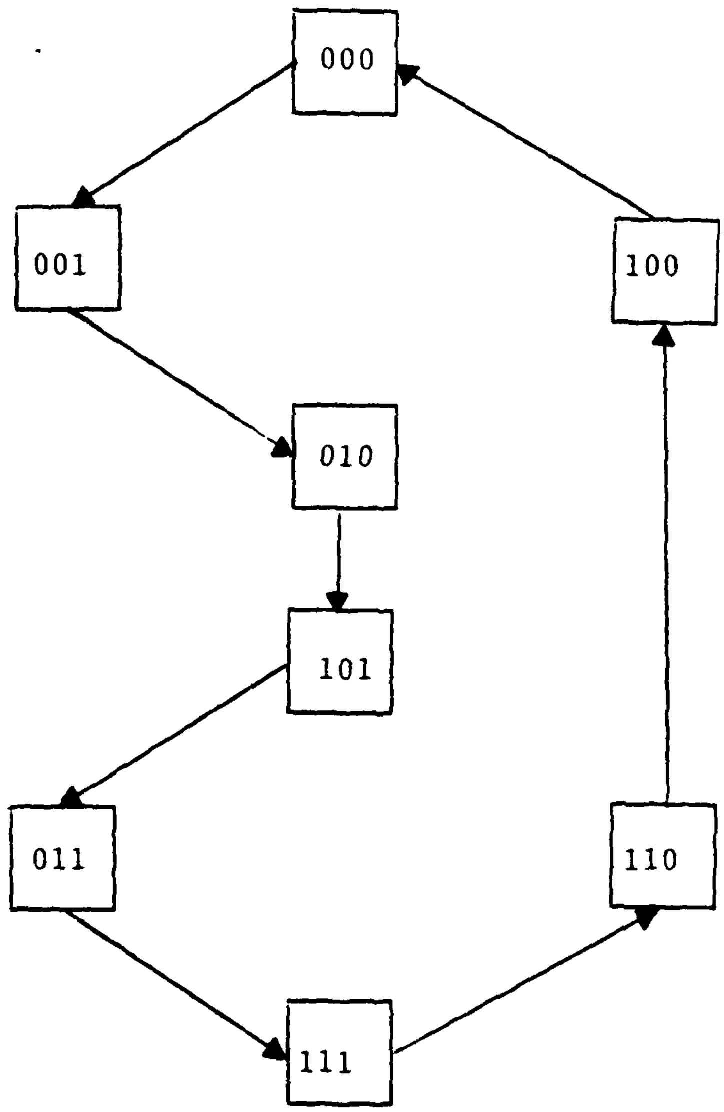


Fig. 4.3 Cyclic Composition of Table 4.4.

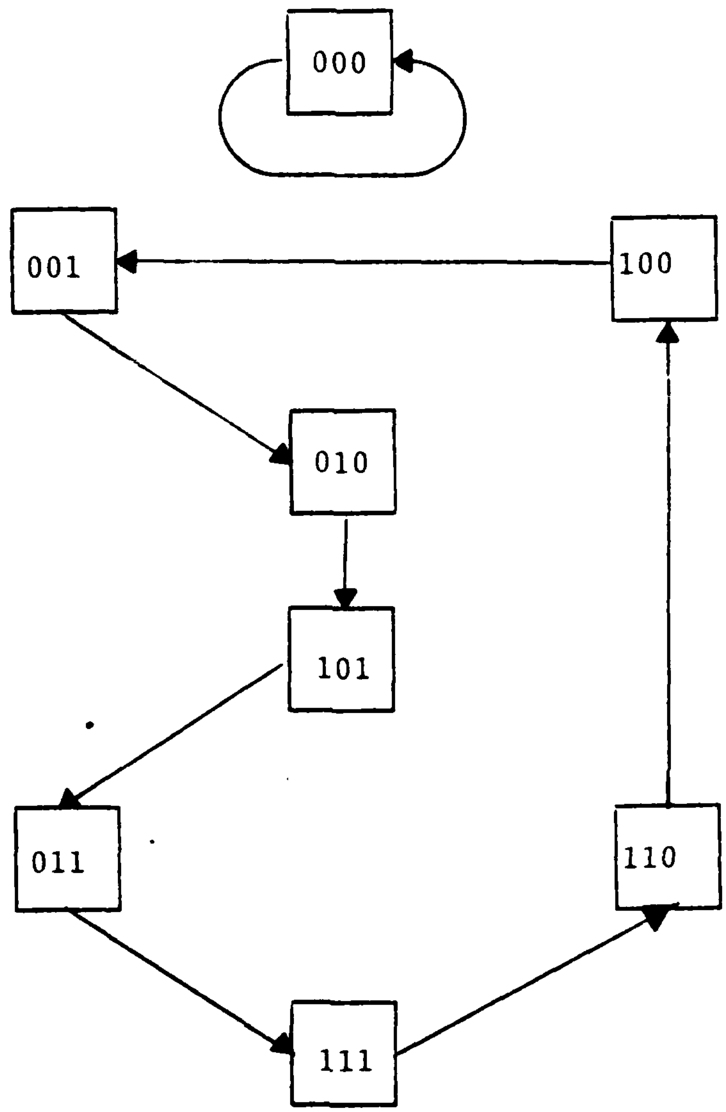


Fig. 4.4 Cyclic Composition of Table 4.7.

This change in the truth table causes the cycle to split into two cycles. Suppose $(x_n \cdots x_1)$ is on element on upper of two cycles as shown in Fig. 4.5 and that its successor is

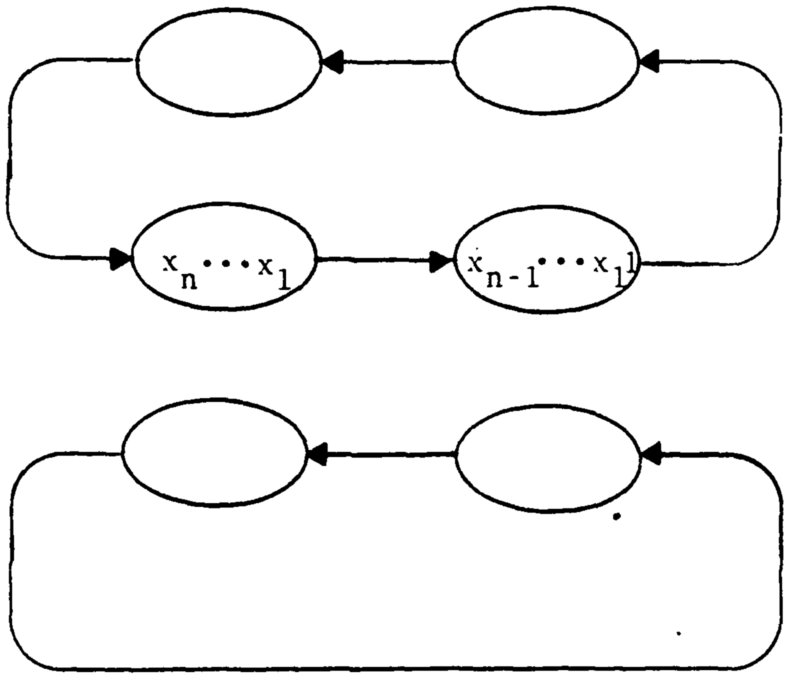


Fig. 4.5 Two Cycle Graph.

$X_{n-1} \cdots x_1$ 1 as $f(x_1 \cdots x_n) = 1$. If the truth table is now changed so that $f(x_1 \cdots x_n) = 0$, then $x_{n-1} \cdots x_1 0$ is the new successor for $x_n \cdots x_1$. Retaining the cycles only condition $f(x_1 \cdots x_{n-1} \overline{x}_n)$ must now be 1. So the successor of $\overline{x}_n x_{n-1} \cdots x_1$ is now $x_{n-1} \cdots x_1 1$. One of two things can happen how depending on whether $x_{n-1} \cdots x_1 0$ is on the upper or lower cycle.

First if ($x_{n-1} \cdots x_1 0$) is on the upper cycle, then that cycle is split as shown in Fig. 4.6 forming three cycles.

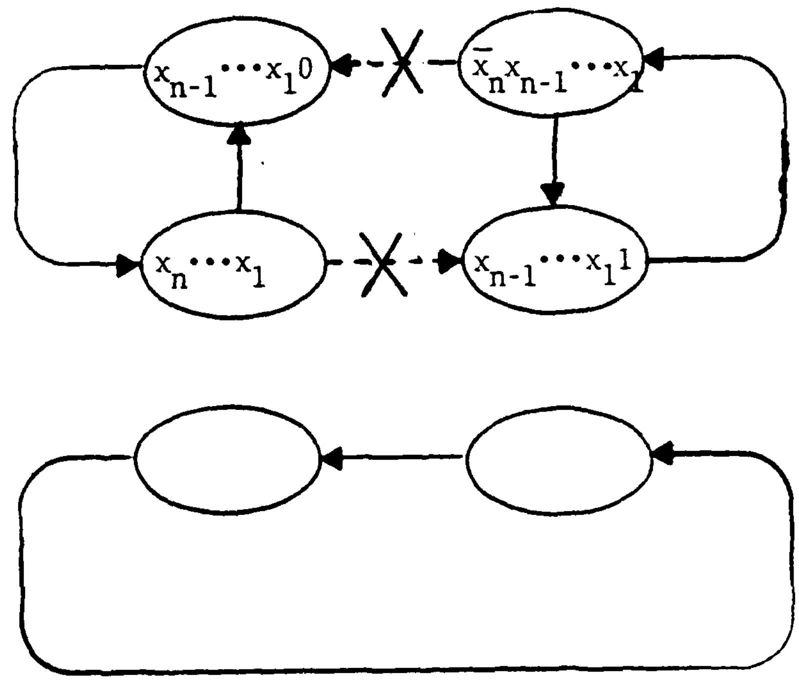


Fig. 4.6 Cycle Splitting.

Otherwise if ($x_{n-1} \cdots x_1 0$) is on the lower cycle, then those two cycles are joined as shown in Fig. 4.7.

Clearly the parity of the cycles changes for each change in the truth table. The following theorem from Golomb [Ref. 7: p. 122] shows that the parity of the number of cycles and the parity of the truth table generator are equal.

<u>Theorem 3</u>. For n>2, the parity of the truth table is equal to the parity of the number of cycles of the truth table.

Since a de Bruijn sequence is one cycle by definition, the following corollary of theorem 3 is given.

Corollary. If a sequence is de Bruijn, then the parity of the TT of that sequence is odd.

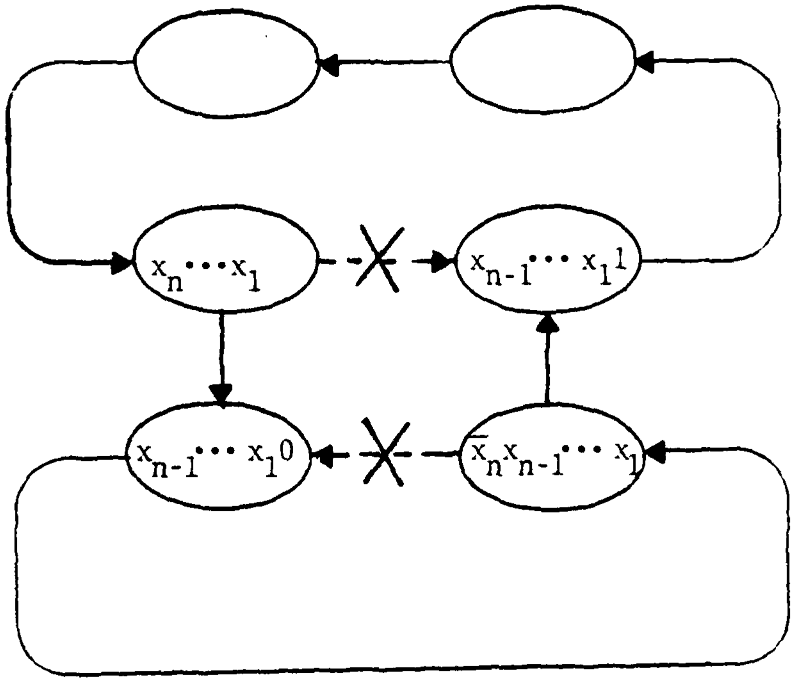


Fig. 4.7 Cycle Joining.

F. SPECIAL NUMBERS

Analyzing the effects of de Bruijn sequence operators on their truth tables, certain binary numbers are found to have special properties which can be exploited in later analysis.

1. Type RC Numbers

If $X = x_1 \cdots x_n$ is a binary number, such that $\bar{x}_n \cdots \bar{x}_1 = x_1 \cdots x_n$ then X is called a RC_n number.

Lemma 4. For n even there are exactly $2^{\frac{n}{2}}$ type RC numbers.

Proof. For n even, if X is an RC number, then $x_1 \cdots x_n = \bar{x}_n \cdots \bar{x}_1$ by definition. Therefore, $x_1 = \bar{x}_n$, $x_2 = \bar{x}_{n-1}$, etc. Thus $x_1 \cdots x_n$ can be written as $x_1 x_2 \cdots x_{\frac{n}{2}} \bar{x}_{\frac{n}{2}} \cdots \bar{x}_2 \bar{x}_1$. Since $\frac{n}{2}$ positions can be filled in either of two ways, the total number of RC numbers for n even is $2^{\frac{n}{2}}$.

Q.E.D.

There are no type RC numbers for n odd, since this would require that $x_{\frac{n+1}{2}} = \bar{x}_{\frac{n+1}{2}}$ which is impossible.

2. Type R Numbers

If $X = x_1 \cdots x_n$ is a binary number such that $x_1 \cdots x_n = x_n \cdots x_1$, then it is called a type R_n number.

Lemma 5. For n even there are exactly $2^{\frac{n}{2}}$ type R numbers, and for n odd there are exactly $2^{\frac{n+1}{2}}$ type R numbers.

Proof. The proof for n even parallels that of Lemma 4 and will not be repeated. For n odd, since $x_{\frac{n+1}{2}} = x_{\frac{n+1}{2}}$ there are $\frac{n+1}{2}$ positions to be filled in either of two ways. Therefore, for n odd there are exactly $2^{\frac{n+1}{2}}$ type R numbers.

Q.E.D.

A useful way of organizing special numbers and other truth table values to assist in evaluating effects on the generators for various operators is shown in Table 4.8 for n = 5 and n = 6. The numbers are the decimal equivalent of the binary (n-1)-tuple, $x_{n-1} \cdots x_1$, from the feedback function

$$x_0 = x_n + g\ (x_1 \cdots x_{n-1})$$

which also corresponds to the truth table state.

For example for n = 5, if f (2) = 1 for some sequence S then: f (11) = 1 for r $\overline{S}$, f (4) = 1 for r S, and f (13) = 1 for $\overline{S}$. In general, for some operator β and some number a, if f (a) = 1 for some sequence S, then f (βa) = 1 for β S. Special numbers have properties in addition to this. If α is a type RC number and f (α) = 1 for S, then f (α)= f (r $\overline{\alpha}$) = 1 for r $\overline{S}$. Similarly if γ is a type R number and f (γ) = 1 for S, then f (γ) = f (rγ) = 1 for r S. Thus, if we want to shown S is a sequence such that S = r $\overline{S}$, then if f (a) = 1 in S, then f (r $\overline{a}$) = 1 in S also.

G. GENERATOR ANALYSIS

Analysis of the generators for the de Bruijn sequences S, r S, $\overline{S}$, & r $\overline{S}$ in Table 4.9 appears to indicate the generator G of sequence $\overline{S}$ denoted G ($\overline{S}$) is the reverse of the generator G (S). Notice also that G (r S) = r [G (r $\overline{S}$)] for S. Although Table 4.9 is an example of a single sequence S, it is not an isolated incident; in general the result G (S) = r [G ($\overline{S}$)] holds and is proven in the following theorem.

TABLE 4.8

TRUTH TABLE ANALYSIS CHART

S	$\bar{S}$	rS	r$\bar{S}$	
$\bar{S}$	S	r$\bar{S}$	rS	
rS	r$\bar{S}$	S	$\bar{S}$	
r$\bar{S}$	rS	$\bar{S}$	S	
0	15	0	15	← Truth table states (n=5)
1	14	8	7	Type R numbers
2	13	4	11	(0,6,9,15)
3	12	12	3	Type RC numbers
5	10	10	5	(3,5,10,12)
6	9	6	9	
0	31	0	31	← Truth table states (n=6)
1	30	16	15	Type R numbers
2	29	8	23	(0,4,10,31)
3	28	24	7	Type RC numbers
4	27	4	27	(None)
5	26	20	11	
6	25	12	19	
9	22	18	13	
10	21	10	21	
14	17	14	17	

TABLE 4.9

GENERATOR ANALYSIS FOR de BRUIJN SEQUENCES

$(r\ S \neq \bar{S})$

S	11111001000111011010011000001010
$\bar{S}$	11111010100000110111000100101100
rS	11111010100000110010110111000100
$r\bar{S}$	11111001101001000111011000001010

	$x_{n-1},\cdots,x_1$	G (S)	G ($\bar{S}$)	G (rS)	G ($r\bar{S}$)
	0	1	1	1	1
	1	0	1	1	0
	2	1	0	0	1
$\boxed{}$	$\boxed{3}$	1	0	0	1
Type RC	4	0	1	1	0
	$\boxed{5}$	0	1	1	0
	6	0	$\underline{1}$	0	1
Type R	7	0	1	1	0
(other than	8	1	0	0	1
0 & 15)	9	1	0	$\underline{1}$	0
	$\boxed{10}$	1	0	0	1
	11	1	0	0	1
	$\boxed{12}$	0	1	1	0
	13	0	1	1	0
	14	1	0	0	1
	15	1	1	1	1

Theorem 4: If S is a de Bruijn sequence with generator G (S), then $\overline{S}$ is a de Bruijn sequence with generator r [G (S)].

Proof: Let G (S) = $\{g_1, g_2, \cdots, g_{2^{n-1}}\}$ then the full truth table for S is given in Table 4.10.

TABLE 4.10

FULL TRUTH TABLE FOR S

x_n	x_{n-1}	x_{n-2}	$\cdots$	x_1	$f(x_1, \cdots, x_n)$
0	0	0	$\cdots$	0	g_1
0	0	0	$\cdots$	1	g_2
$\vdots$				$\vdots$	$\vdots$
0	1	1	$\cdots$	1	$g_{2^{n-1}}$
1	0	0	$\cdots$	0	$\overline{g}_1$
1	0	0	$\cdots$	1	$\overline{g}_2$
$\vdots$					$\vdots$
1	1	1	$\cdots$	1	$\overline{g}_{2^{n-1}}$.

If $x_n, x_{n-1}, \cdots, x_1, f(x_1 \cdots x_n)$ is a (n+1) long string in S, then $\overline{x}_n, \overline{x}_{n-1}, \cdots, \overline{x}_1, \overline{f(x_1 x_2 \cdots x_n)}$ is a (n+1) long string in $\overline{S}$.

Therefore we can transform each line of the full truth table for S by taking its complement to get Table 4.11.

TABLE 4.11

FULL TRUTH TABLE FOR $\overline{S}$ (NON STANDARD)

x_n	x_{n-1}	x_{n-2}	$\cdots$	x_1	$f(x_1, x_2, \cdots, x_n)$
1	1	1	$\cdots$	1	$\overline{g}_1$
1	1	1	$\cdots$	0	$\overline{g}_2$
$\vdots$			$\vdots\vdots$		$\vdots$
1	0	0	$\cdots$	0	$\overline{g}_{2^{n-1}}$
0	1	1	$\cdots$	1	g_1
0	1	1	$\cdots$	0	g_2
$\vdots$			$\vdots\vdots$		$\vdots$
0	0	0	$\cdots$	0	$g_{2^{n-1}}$

Rearranging the rows from top to bottom, the TT for $\overline{S}$ is placed in standard form as depicted in the Table 4.12. Further examination of the generators in Table 4.9 shows the structure of G $(\overline{S})$ to be remarkably similar to the structure of G (r S), with the exception that the functional values of type R numbers (6 & 9) are transposed. If, however, these values should be the same then clearly G $(\overline{S})$ = G (r S) or S = r $\overline{S}$. Indeed, this occurs twice for n = 3 and 64 times for n = 5, and will be shown to happen for all odd $n \geq 3$. This cannot happen for n even, because there are no type RC numbers in the truth table.

TABLE 4.12

FULL TRUTH TABLE FOR $\bar{S}$

x_n	x_{n-1}	x_{n-2}	$\cdots$	x_1	$f(x_1, x_2, \cdots, x_n)$
0	0	0	$\cdots$	0	$g_{2^{n-1}}$
$\vdots$			$\vdots$		$\vdots$
0	1	1	$\cdots$	0	g_2
0	1	1	$\cdots$	1	g_1
1	0	0	$\cdots$	0	$\bar{g}_{2^{n-1}}$
$\vdots$			$\vdots$		$\vdots$
1	1	1	$\cdots$	0	$\bar{g}_2$
1	1	1	$\cdots$	1	$\bar{g}_1$

Clearly G $(\bar{S})$ is r [G (S)]

Q.E.D.

V. RESULTS

Examining Table 3.2 it can be quickly determined that the number of de Bruijn sequences of given complexity for $3 \leq n \leq 6$ is 0 (mod 4). Games and Chan [Ref. 1] made this conjecture, but were able to show only that the distribution α (c, n) was 0 mod 2. Their result followed by showing that for a given de Bruijn sequence S of complexity c, there existed a unique de Bruijn sequence $\bar{S}$ also having complexity c. In Section III it was shown that reverse and reverse dual operators preserve the complexity of de Bruijn sequences as well.

If each of these new operators produced unique de Bruijn sequences, then it would be easy to show that the numbers $\alpha (c, n) \equiv 0 \pmod 4$ since for every de Bruijn sequence S there would also exist $\bar{S}$, r S and r $\bar{S}$ all de Bruijn and unique having complexity c. As mentioned in Section IV, the sequences r $\bar{S}$ and r S are not always distinct from the sequences S & $\bar{S}$ respectively; however, for n even the next theorem shows that all four of these sequences are distinct. In addition it will be shown that RC sequences occur for all n odd, and a lower bound for the number of RC sequences will be established for all n and investigated for n = 7.

A. α (c, n) $\equiv$ 0 MODULO 4 FOR n EVEN

Etzion and Lempel [Ref. 6] give a proof that α (c, n) is congruent to 0 mod 4 for c n even. In the proof for n even given here a different approach is taken.

Theorem 5. For even $n \geq 3$, the numbers $\alpha (c, n) \equiv 0 \pmod 4$ for a de Bruijn sequence S of span n and complexity c.

Proof. From theorem 1, it is known that for a de Bruijn sequence S of complexity c, that $\bar{S}$, r S & r $\bar{S}$ also have complexity c. Games and Chan showed the uniqueness of $\bar{S}$, so all that remains is to show that r S is distinct from S, or equivalently that r $\bar{S}$ is distinct from S for n even.

Suppose on the contrary that S = r $\bar{S}$. Then consider an n-tuple $0a_{n-1} \cdots a_1$ in the truth table such that $f (a_1 \cdots a_{n-1} 0) = 1$. Then $0a_{n-1} \cdots a_1 1$ is a n+1 long string in S, and $o\ \bar{a}_1 \cdots \bar{a}_{n-1} 1$ is a n+1 long string in r $\bar{S}$. If S = r $\bar{S}$
then $0\ \bar{a}_1 \cdots \bar{a}_{n-1}1$ is a n+1 long string in S,
which implies
that $1\ \bar{a}_1 \cdots \bar{a}_{n-1}\ 0$ is a n+1 long string in S, and therefore $f (\bar{a}_1 \cdots \bar{a}_{n-1} 0) = 1$. This must hold for every $x_1 \cdots x_{n-1} 0$ such that $f (x_1 \cdots x_{n-1} 0) = 1$. Pairing these vectors, $a_1 \cdots a_{n-1}$ and $\bar{a}_{n-1} \cdots \bar{a}_1$, two at a time results in an even number of 1's in the truth table. So there must be a vector which pairs with itself as the parity of ones is odd. But then $x_{n-1} \cdots x_1 = \bar{x}_1 \cdots \bar{x}_{n-1}$ is a

contradiction since $\overline{x}_{\frac{n+1}{2}} \neq x_{\frac{n+1}{2}}$ for n even. Therefore for n even $S \neq r\,\overline{S}$ and $r\,S \neq \overline{S}$, so $r\,S$ and $r\,\overline{S}$ are distinct and $\alpha\,(c,\, n) \equiv 0 \bmod 4$.

Q.E.D.

B. RC SEQUENCES EXIST FOR ALL ODD $n \geq 3$

The results $\alpha(c,n) \equiv 0 \pmod 4$ for all n are not obtained as hoped for. For $n = 3$ and $n = 5$ RC sequences are known to exist, but it is not known if this is true for all odd $n \geq 3$ or just some. The following theorem shows that RC sequences exist for all odd $n \geq 3$. Recall that RC sequences are those sequences S for which $S = r\,\overline{S}$ or equivalent by $\overline{S} = r\,S$ since $r\,(S) = r\,(r\,\overline{S})$ yields $\overline{S} = r\,S$.

Lemma 6. The sequence formed by joining all pure cycles of n-tuples is de Bruijn.

Proof. The least elements of each pure cycle for a given n are easily composed and arranged in order by weight according to the algorithm in Appendix A. Since the pure cycles contain all n-tuples exactly once, all that remains is to show how each pure cycle can be interconnected, thus forming a de Bruijn sequence. For each cycle of wt $a \geq 1$ there exists a least element "2X+1" (necessarily odd, if it were even (2X) then "X" is on the same cycle and smaller) whose predecessor "X" has wt (a-1). Since the pure cycles are ordered by weight, it is clear that the least element of each pure cycle of wt $a \geq 1$ can be joined to a cycle of wt (a-1) by changing the truth table for its predecessor, i.e. g(X) becomes 1. In this way all the pure cycles are now interconnected, and the sequence is de Bruijn.

Q.E.D.

Theorem 6. For all odd $n \geq 3$, there exist RC de Bruijn sequences.

Proof. An RC de Bruijn sequence can always be composed in the following way for $n \geq 3$. First construct a light cycle (LC) for those pure cycles of wt $a \leq \frac{n-1}{2}$ identical to the method used in Lemma 6. Note that when the cycles of wt $\frac{n-1}{2}$ are joined to cycles of smaller weight the truth table is changed at positions whose weight is less than $\frac{n-1}{2}$. For every X such that f (X) = 1 in LC, induce f $(r\,\overline{X})$ = 1 on the pure cycles of wt $(n-a) \geq \frac{n+1}{2}$, forming a heavy cycle (HC). Note that HC is the reverse complement of LC. All of the positions changed so that g (X) = 1 on HC are of weights bigger than or equal to $\frac{n+1}{2}$.

Join LC and HC by using a type RC number, y, which exist by Lemma 4, since for n odd the truth table states have even (n-1) length. The RC number y has weight $\frac{n-1}{2}$. Letting f (Y) = 1 then joins LC and HC forming one sequence. By Theorem 1 the sequence is RC, and by Lemma 6 the sequence is de Bruijn.

Q.E.D.

C. LOWER BOUND ON NUMBER OF RC SEQUENCES

1. Theorem

A weak lower bound on the number of RC de Bruijn sequences is established.

Theorem 7: A lower bound on the number of de Bruijn RC sequences for odd $n \geq 3$ is $A \cdot 2^{\frac{n-1}{2}}$, where A is the number of possible different interconnections for pure cycles of weight $\leq \frac{n-1}{2}$ as determined by the Best method [Ref. 5].

Proof: Restricting attention to those cycles of wt $\leq \frac{n-1}{2}$ (since the other connections are induced), the Best method gives the number A of possible different interconnections for the pure cycles of weight $\leq \frac{n-1}{2}$. Since there exist $2^{\frac{n-1}{2}}$ type RC numbers, this gives $2^{\frac{n-1}{2}}$ ways of joining A different pairs of cycles for a total of $A \cdot 2^{\frac{n-1}{2}}$ possibilities.

Q.E.D.

2. Example for n = 7

The pure cycles of wt ≤ 3 and the adjacency matrix of A are given in Tables 5.3 and 5.4 respectively. The value of the determinant of the adjacency matrix is 38,880 for any diagonal element. This gives a lower bound of 38,880 X 8 = 311,040 for the number of de Bruijn RC sequences.

TABLE 5.1

PURE CYCLES FOR N=7 & wt $\leq$ 3

0						
1	2	4	8	16	32	64
3	6	12	24	48	96	65
5	10	20	40	80	33	66
7	14	28	56	112	97	67
9	18	36	72	17	34	68
11	22	44	88	49	98	69
13	26	52	104	81	35	70
19	38	76	25	50	100	73
21	42	84	41	82	37	74

TABLE 5.2

ADJACENCY MATRIX FOR n=7

1	-1	0	0	0	0	0	0	0	0
-1	7	-2	-2	-2	0	0	0	0	0
0	-2	7	0	0	-2	-1	-1	-1	0
0	-2	0	7	0	-1	-1	-1	0	-2
0	-2	0	0	7	0	-1	-1	-2	-1
0	0	-2	-1	0	3	0	0	0	0
0	0	-1	-1	-1	0	3	0	0	0
0	0	-1	-1	-1	0	0	3	0	0
0	0	-1	0	-2	0	0	0	3	0
0	0	0	-2	-1	0	0	0	0	3

VI. CONJECTURES

Although a complete proof for α (c, n) $\equiv$ 0 (mod 4) has yet to be given, a technique for grouping RC sequences in a 0 (mod 4) fashion will be demonstrated for n = 5. However, the technique fails to preserve complexity in every case. However, for n = 5 the complexitites produced were congruent to 0 (mod 2) and when the complement of each sequence is added a 0 (mod 4) distribution results. It remains, however, that the complexity of the generated sequences are not determined apriori. The technique presented is interesting in its own right, and it is hoped that the interested reader may be able to apply it towards a solution of the 0 (mod 4) distribution.

A. RC SEQUENCE GENERATOR TECHNIQUE

Table 6.1 shows a listing of four RC sequences and their generators. The technique used was to choose in turn each of the type RC numbers to have a functional value of 1.

Note that the complexity of S_3 and S_5 are both 23, while the complexity of S_{10} and S_{12} are both 29. At present there is no way of knowing which pair of RC sequences will have the same complexity, except that it appears $C(S_k)=C(S_{r\bar{K}})$ iffall four RC sequences have the same complexity. A listing of the generators for each of the 32 pairs of RC sequences

TABLE 6.1

RC GENERATOR TECHNIQUE

	$C(S_3)=23$	$C(S_5)=23$	$C(S_{10})=29$	$C(S_{12})=29$
n	$G(S_3)$	$G(S_5)$	$G(S_{10})$	$G(S_{12})$
0	1	1	1	1
1	1	1	1	1
2	0	0	0	0
$\boxed{3}$	$\underline{1}$	0	0	0
4	1	1	1	1
$\boxed{5}$	0	$\underline{1}$	0	0
6	1	1	1	1
7	1	1	1	1
8	1	1	1	1
9	1	1	1	1
$\boxed{10}$	0	0	$\underline{1}$	0
11	0	0	0	0
$\boxed{12}$	0	0	0	$\underline{1}$
13	1	1	1	1
14	1	1	1	1
15	1	1	1	1

$\boxed{}$ indicates RC number.

S_3 = 11111001010001001101110101100000

S_5 = 11111001011000001101110101000100

S_{10}= 11111001010110000011011101000100

S_{12}= 11111000001101110101100101000100 is prepared in Table 6.2. These are listed in order by $W(G)$ and then by complexity as far as the groupings allow. Keep in mind, that the complement of each RC sequence is not included in the table. A close analysis of this table then shows a 0 (mod 4) grouping for the number of RC sequences of complexity c and span n = 5.

B. PROBLEMS

For n = 7, there exist 6 type R numbers (other than 0 & 64) and 8 type RC numbers in the truth table. This large number of possibilities will create considerable problems in extending this idea to n = 7 and beyond.

Using this approach a large scale computer based analysis is essential to any further investigation of RC sequences for $n \geq 7$. Considering the number of de Bruijn sequences for n = 7 to be 2^{57} or 1.42×10^{17} an algorithm for generating only the RC sequences is crucial, since generating 10^{17} de Bruijn sequences is not technically feasible. Though Theorem 7 establishes the lower bound of 311,040 RC sequences for n = 7, this is a very weak bound and a presumably still conservative estimate would be on the order of 10^6 actual RC sequences for n = 7.

TABLE 6.2

RC de BRUIJN SEQUENCE 0 (Mod 4)
GROUPING FOR n=5

```
              G (S)                          W[G(S)]     C(S)
100  0  1  1  1001  0  0  1  101                7          25
           0           1                        7          25
           1           0                        7          27
     1     0                                    7          27
100  0  0  0  1011  1  0  0  011                7          27
                    0     1                     7          27
     1                    0                     7          29
     0     1                                    7          29
101  0  0  0  0010  0  1  1  011                7          31
                    1     0                     7          31
           1        0                           7          31
     1     0                                    7          31
100  0  1  0  0010  0  0  1  111                7          31
                    1     0                     7          31
           1        0                           7          31
     1     0                                    7          31
101  0  0  0  1011  1  1  0  011                9          31
                    0     1                     9          31
     1                    0                     9          31
     0     1                                    9          31
100  0  1  0  1011  1  0  0  111                9          31
                    0     1                     9          31
     1                    0                     9          31
     0     1                                    9          31
111  0  0  0  1111  1  1  0  011               11          23
                    0     1                    11          23
     1                    0                    11          29
     0     1                                   11          29
101  0  1  1  1011  0  1  0  111               11          25
           0              1                    11          25
     1                    0                    11          27
     0              1                          11          27
```

APPENDIX A

ALGORITHM FOR GENERATING LEAST ELEMENT FOR EVERY PURE CYCLE

This algorithm is designed to produce the least element of each pure cycle. The least elements are initially in decreasing order. A simple rearrangement will group the least elements by weight in decreasing order if necessary. The algorithm to be presented is an adaptation of the θ-operation discussed in a paper by Fredricksen and Maiorana [Ref. 10], to generate a lexicographic list of necklaces.

ALGORITHM: Begin the Ω operation with the zero n-tuple, which is the first least element for the zero pure cycle.

Ω Operation: $\Omega\ (x_1 \cdots x_n) = (y_1 \cdots y_n) = Y$

1. Find the largest subscript j such that

   $x_j = 0$ and $x_k = 1$ for $k>j$.

2. Form $x_1\ x_2 \cdots x_{j-1}\ 1$ where $x_j = 1$
3. Repeat $x_1\ x_2 \cdots x_{j-1}1$ until n numbers are produced.

   i) If $n = tj$ for some integer t then let

   $Y=x_1\cdots x_{j-1}1x_1\cdots x_{j-1}1\cdots x_1\cdots x_{j-1}1$

   ii) If $n>tj$ then finish with $x_1\cdots x_{n-t_j}$,

   then let $Y=x_1\cdots x_{j-1}1\cdots x_1\cdots x_{j-1}1x_1\cdots x_{n-t_j}$.

4. Y is a least element iff $n=tj$
5. Repeat beginning at step 1 until the final least element $(1)^n$ is reached.

TABLE A.1

Ω ALGORITHM FOR N = 7

		Wt
0000000		0
0000001		1
0000010	*	
0000011		2
0000100	*	
0000101		2
0000110	*	
0000111		3
0001000	*	
0001001		2
0001010	*	
0001011		3
0001100	*	
0001101		3
0001110	*	
0001111		4
0010010	*	
0010011		3
0010100	*	
0010101		3
0010110	*	
0010111		4
0011001	*	
0011010	*	
0011011		4
0011100	*	
0011101		4
0011110	*	
0011111		5
0101010	*	
0101011		4
0101101	*	
0101110	*	
0101111		5
0110110	*	
0110111		5
0111011	*	
0111101	*	
0111110	*	
0111111		6
1111111		7

* Unacceptable

Table A.1 illustrates the algorithm for n=7. Arranging in order by weight, the least element for each pure cycle is tabulated in Table A.2.

TABLE A.2

PURE CYCLE LEAST ELEMENTS (n=7)

0	0000000	0
1	0000001	1
2	0000011	3
	0000101	5
	0001001	9
3	0000111	7
	0001011	11
	0001101	13
	0010011	19
	0010101	21
4	0001111	15
	0010111	23
	0011011	27
	0011101	29
	0101011	43
5	0011111	31
	0101111	47
	0110111	55
6	0111111	63
7	1111111	127

LIST OF REFERENCES

1. Chan, A. H., Games, R.A., and Key, E.L., "On the Complexities of de Bruijn Sequences", Journal of Combinatorial Theory, Series A, Vol. 33, pp. 233-246, November 1982.

2. Birkoff, G. and Bartee, T.C., Modern Applied Algebra, McGraw-Hill, 1970.

3. Golomb, S., Digital Communications with Space Applications, Prentice-Hall, 1964.

4. Hall, M. Jr., Combinatorial Theory, Blaisdell, 1967.

5. Fredricksen, H. M., "A Survey of Full Length Nonlinear Shift Register Cycle Algorithms", SIAM Review, Vol. 24, pp. 195-221, April 1982.

6. Technion Report 268, On the Distribution of de Bruijn Sequences of Given Complexity, by T. Etizon and A. Lempel, February 1983.

7. Golomb, S., Shift Register Sequences, Holden-Day, 1967.

8. IEEE Proceedings International Information Theory Conference, 1982, On the Complexity of Functions of Linear Shift Register Sequences, by T. Herlestam, p. 166.

9. Games, R.A. and Chan, A.H., "A Fast Algorithm for Determining the Complexity of a Binary Sequence with a Period 2^n", IEEE Trans. on Inform. Theory, January 1983.

10. Fredricksen, H.M. and Maiorana, J., "Necklaces of Beads in k Colors and k-ary de Bruijn Sequences", Discrete Math., Vol. 23, No. 3, September 1978.

INITIAL DISTRIBUTION LIST

		No. Copies
1.	Defense Technical Information Center Cameron Station Alexandria, Virginia 22314	2
2.	Library, Code 0142 Naval Postgraduate School Monterey, California 93940	2
3.	Department Chairman, Code 53Wm Department of Mathematics Naval Postgraduate School Monterey, California 93940	1
4.	Prof. Fredricksen, Code 53Fs Department of Mathematics Naval Postgraduate School Monterey, California 93940	5
5.	Major Robert L. Holdahl, USMC Department of Mathematics United States Naval Academy Annapolis, Maryland 21402	6

END

FILMED

6-84

DTIC